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1982 J. Phys. A: Math. Gen. 15 409

(http://iopscience.iop.org/0305-4470/15/2/014)

J. Phys. A: Math. Gen. 15 (1982) 409–427. Printed in Great Britain

Motion of a classical particle with spin: I. The canonical theory of multipliers

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Received 27 January 1981, in final form 3 August 1981

Abstract. Using Dirac's theory of canonical multipliers (as modified by Shanmugadhasan), we investigate and derive the complete canonical formalism for a well known relativistic model of a classical spinning particle (generalised to include asymmetry). While the procedure is well known from existing dynamical theory, its application to a relativistic asymmetric particle or top has not before been attempted. In the past, *ad hoc* means have been found for the formulation of a Hamiltonian method, and these have led to incomplete pictures for the model considered. Our formalism restates many well known results, using the calculus of Dirac brackets, but we also derive some new results. In paper II we use a computer to evaluate Dirac brackets and thus sidestep the task of algebraic calculation. Without such help the complete problem appears to be intractable. Paper I deals with the problem in general and explains the calculation of the complete set of constraints, the multipliers and the Hamiltonian equations for the model.

1. Introduction

In a previous paper (Ellis 1981) we justified the use of the 'proper time derivative' of operators in the theory of the Dirac equation in quantum mechanics, based on an equation of motion similar to the Heisenberg equation for the coordinate time rate. The formalism had certain similarities to the classical formalism for a free particle, using proper time rather than coordinate time as the independent variable. In the present work we shall not dwell on the quantum mechanical problem but consider the classical formalism, which is the other side of this analogy.

Corben (1968, §§ 6–8) draws some detailed comparisons between the helical solutions of the classical free particle equations

$$\dot{p}^{\mu}=0, \qquad \dot{s}^{\mu\nu}+2p^{[\mu}\dot{x}^{\nu]}=0, \qquad s^{\mu\nu}\dot{x}_{\nu}=0 \tag{1.1}$$

(where a dot denotes differentiation with respect to the proper time) and the zitterbewegung of the Dirac equation in quantum mechanics. In these comparisons the classical canonical formalism lacks a concrete basis. The 'Hamiltonian' is postulated and the 'Poisson bracket' relations are obtained not by the use of the usual definition but from known properties of the observables (e.g. the Poisson bracket relations for the spin tensor $s^{\mu\nu}$ are based on those satisfied by the infinitesimal generators of the inhomogeneous Lorentz group). In the following we remove this restriction and give the

† This work was carried out in part while on leave of absence between October 1979 and September 1980.

0305-4470/82/020409+19$02.00

complete canonical description for these equations which, we believe, has not before been attempted†. The association with the quantum mechanical equations (based on the use of the above-mentioned operator) can be investigated more precisely than has been the case in Corben's theory.

A crucial feature of the equations (1.1) (we shall refer to these equations of Corben subsequently as the 'free spinning particle equations') is that in the canonical formalism the third member of the equations (1.1)—the 'supplementary condition'—leads to constraints between the coordinates and the momenta. However we try to represent the spin tensor in terms of coordinates and momenta in the canonical formalism, this third equation will lead to constraints that cannot be resolved. Consequently we must use the canonical formalism introduced by Dirac (1964, see below) in order to take these constraints into account properly. The revised canonical formalism leads to 'modified' or 'Dirac' brackets in place of Poisson brackets, and in II we show that many of the 'Poisson bracket relations' postulated by Corben for these classical free spinning particle equations can be confirmed in our formalism as Dirac brackets. (Similar work by Rafanelli (1967) is also confirmed in this sense.)

We shall postulate a simple Lagrangian function for Corben's model (slightly generalised to include asymmetric particles), and we shall subsequently make use only of the geometry of a moving tetrad of vectors in specal relativity. By such means we calculate without further postulates the complete set of canonical constraints, the canonical multipliers and the covariant Hamilton equations (in I), and the Dirac brackets and the relations they satisfy (in II). The choice of this Lagrangian is therefore important. The one given here is the most strongly related to the ordinary classical formalism in having an identifiable 'spin-energy' term, from which the more general equations than Corben's—equations for a particle endowed with moments of inertia I_{ij}—can be obtained. Corben relaxes the supplementary condition (Corben 1968, § 9) when describing this more general case, and his theory is developed by analogy with non-relativistic theory. He splits the spin tensor into components that generalise the classical equations. Obviously it is better to describe the motion of the symmetric and asymmetric cases entirely within the previous framework where the supplementary condition is used. Our Lagrangian equations, which include a more detailed structure of the spin tensor and which lead to the same basic equations of motion subject to the Frenkel condition as a supplementary condition, have already been discussed (Ellis 1975a) and we shall not reconsider them in great detail.

As in the present problem, the occurrence of subsidiary conditions between the dynamical variables leads us to use Lagrange multipliers in the Lagrangian function‡. The resulting Lagrangian is then degenerate (i.e. its Hessian matrix is of constant singular rank everywhere within the space of the arguments of the Lagrangian), with the consequence that constraints between the *canonical* variables arise. This, in turn, leads to the need for a modified canonical theory. Such a canonical theory was developed originally by Dirac (1950, 1959, 1964, 1969), who also included a quantisation procedure. The method has been described in several texts (see, for example, those of

† It appears that the complete formalism for equations of this type has not before been given, apart from two well known instances: that for the spherical particle based on the Nakano condition was considered recently by Hanson and Regge (1974), and Dirac's formalism has also been used by Hughes (1961).

‡ In essence, this use of Lagrange multipliers is an *essential* one for the present Lagrangian formalism, since no equivalent formalism based on a Lagrangian homogeneous of the first order in the generalised velocities exists, i.e. the variation of $\int L\, d\tau$ cannot be simplified by change of independent variable (cf Barut 1964, pp 60–5).

Mercier (1959), Sudarshan and Mukunda (1974), Mann (1974)), and in the literature the method has been developed and applied by a number of authors (Anderson and Bergmann 1951, Haag 1952, Kundt 1966, Shanmugadhasan 1963, 1973, Mukunda 1976, and many others). This canonical theory leads to the definition of the Dirac bracket, which is the natural generalisation of the Poisson bracket, to be adopted in the Dirac correspondence when there exist independent constraints between the phase space variables.

Shanmugadhasan (1973) in particular has shown that where there are first-order Lagrange equations, Dirac's multiplier rule is inadequate. The evaluation of the canonical multipliers in this particular problem reaffirms Shanmugadhasan's finding, and we also show by this example that his procedure is itself *insufficient* for the determination of all the multipliers, since his method applies only to Lagrangian systems for which all of the derivatives of the first-order Lagrange equations hold by virtue of all the undifferentiated Lagrange equations. Many Lagrangian systems (including the present) whose degeneracy arises by the use of Lagrange multipliers do not fulfil these conditions and some extension of his method is required. The present problem is sufficiently complicated that the generality of our approach is well displayed.

In § 2 we introduce the Lagrangian formalism by defining the geometrical variables involved. In § 3 we describe the method by which we calculate all the canonical constraints without the use of the multiplier rule. These constraints arise in three ways as first- and second-kind primary constraints and also as 'secondary constraints' (the latter form the basis of the extension to Shanmugadhasan's theory). The mutual Poisson brackets of all these constraints (called the set of subsidiary conditions on the canonical variables) are evaluated and simplified by using the constraints, and we distinguish between constraints that are first and second class. In contrast to the iterative use of the multiplier rule (Dirac 1964), which we have found to be inapplicable to this type of problem, we use in § 4 the multiplier rule only *once* to confirm that we have all the right equations. This entails adding, to the primitive Hamiltonian, canonical multiplier terms for the second-class constraints, which have a physical effect. As in the current theory of canonical multipliers, the number of second-class constraints must be even, and all these multipliers can be determined. The resulting first-class Hamiltonian may be used in Hamilton's equations and these equations give all the correct equations, confirming that we have found a complete set of constraints.

2. Lagrangian formalism

2.1. Geometrical notions

We initially suppose that the particle's position relative to some fixed inertial reference system is given by the coordinates x^μ $(\mu = 0, 1, 2, 3)$ where $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$.

The metric of special relativity is taken in the form $d\tau^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$ with $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. A dot denotes differentiation with respect to the proper time τ, and square brackets denote antisymmetrisation: thus, $2a^{[\mu}b^{\nu]} = a^\mu b^\nu - a^\nu b^\mu$. The relative tensors (alternating symbols) $\varepsilon_{\mu\nu\alpha\beta}$, $\varepsilon^{\mu\nu\alpha\beta}$ have the value 1 when the suffixes represent an even permutation of 0, 1, 2, 3; -1 when they represent an odd permutation; 0 otherwise. (The ε-symbol effectively changes sign whenever all of its suffixes are raised or lowered.) The three-suffix symbol ε_{ijk} will also be used. Latin suffixes will range over the values 1, 2, 3; Greek suffixes over 0, 1, 2, 3; repeated suffixes will always be summed (there will be no exceptions), be they Latin or Greek. Duality is defined by

$$s_{\mu^*\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}s^{\alpha\beta}, \qquad s^{\mu^*\nu} = -\tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}s_{\alpha\beta}.$$

The four-velocity $\dot{x}^{\mu}$ satisfies the equation $\dot{x}^{\mu}\dot{x}_{\mu} = 1$, and three internal space-like four-vectors $u^{\mu}_{(i)}$ $(i = 1, 2, 3)$ whose components satisfy the equations

$$\dot{x}^{\mu}u_{(i)\mu} = 0, \qquad u^{\mu}_{(i)}u_{(j)\mu} = -\delta_{ij} \tag{2.1}$$

are called the *internal coordinates* of the particle. For each i the four-vector representing the ith axis is orthogonal to $\dot{x}^{\mu}$, and the three internal axes form, with $\dot{x}^{\mu}$, the *moving tetrad of vectors* representing the instantaneous rest-system for the particle.

In order to measure the rate of rotation of this frame the following three *Lorentz invariant* scalars

$$\omega_{(i)} \overset{\text{def}}{=} -\tfrac{1}{2}c\varepsilon_{ijk}\dot{u}^{\sigma}_{(j)}u_{(k)\sigma} \tag{2.2}$$

are introduced, analogous to the three non-relativistic components of angular velocity $(\omega_x, \omega_y, \omega_z)$:

$$\omega_x = \frac{1}{2}\left(\frac{\mathrm{d}\boldsymbol{j}}{\mathrm{d}t}\cdot\boldsymbol{k} - \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t}\cdot\boldsymbol{j}\right), \ldots,$$

for a moving frame $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ in classical mechanics. This 'point of contact' with classical mechanics is, however, temporarily lost when we try to construct an *angular velocity four-vector* by using relations (2.1). We have

$$\omega^{\mu} \overset{\text{def}}{=} \omega_{(i)}u^{\mu}_{(i)} = \tfrac{1}{2}c\varepsilon^{\mu\lambda\sigma\tau}\dot{u}_{(i)\lambda}u_{(i)\sigma}\dot{x}_{\tau}. \tag{2.3}$$

(We have used the relations (2.1) in a tensor form analogous to $\boldsymbol{i} = \boldsymbol{j}\wedge\boldsymbol{k}, \ldots$ for the moving frame—see § A1.1.)

When the three scalars $\omega_{(i)}$ vanish, the internal velocities $\dot{u}^{\mu}_{(i)}$ do not vanish unless the particle moves with uniform velocity: the three internal axes undergo Thomas motion along the world line (see below):

$$\dot{u}^{\mu}_{(i)} = -(\ddot{x}^{\nu}u_{(i)\nu})\dot{x}^{\mu}.$$

In other words, the u's are Fermi–Walker transported along it. This covariant equation arises in the discussion of the Thomas precession of any space axis (see, for example, Mann (1974, formula (1.104), p 21)). More generally, when $\omega_{(i)} \neq 0$, the kinematical equations (2.2), (2.3) lead via (2.1) to 'extra' ('Thomas') terms in the equation for the rate $\dot{u}^{\mu}_{(i)}$ that are additional to those involving ordinary rotation and which are consistent with the rate of Thomas precession in the absence of rotation (Bacry 1962, Nodvik 1964):

$$\dot{u}^{\mu}_{(i)} = \Omega^{\mu\nu}u_{(i)\nu},$$
$$\Omega^{\mu\nu} \overset{\text{def}}{=} (\ddot{x}^{\mu}\dot{x}^{\nu} - \ddot{x}^{\nu}\dot{x}^{\mu}) + c^{-1}\varepsilon^{\mu\nu\alpha\beta}\omega_{\alpha}\dot{x}_{\beta}. \tag{2.4a}$$

These kinematical equations are obtained from (2.3) by using the derivatives of (2.1) (§ A1.2). The equations (2.4a) can also be expressed in the form

$$\dot{u}^{\mu}_{(i)} = (-\ddot{x}^{\nu}u_{(i)\nu})\dot{x}^{\mu} - c^{-1}\varepsilon_{ijk}\omega_{(j)}u^{\mu}_{(k)}, \tag{2.4b}$$

analogous to the classical results $\mathrm{d}\boldsymbol{i}/\mathrm{d}t = \omega_z\boldsymbol{j} - \omega_y\boldsymbol{k}, \ldots$, but these results include the extra Thomas terms.

Either (2.4*a*) or (2.4*b*) will confirm that when the three scalars $\omega_{(i)}$ vanish (or, equivalently, $\omega^{\mu}=0$), the internal coordinates undergo Thomas motion (the equations are those of Fermi–Walker transport). Conversely, when the rest frame performs this motion, the ω's vanish by (2.2).

We postulate a mechanical spin angular momentum of the particle by direct analogy with classical mechanics:

$$s_{(i)}^{\text{mech}} \stackrel{\text{def}}{=} I_{ij}\omega_{(j)} \qquad (I_{ij}=I_{ji}). \tag{2.5}$$

The I's are constants, or are arbitrary Lorentz invariant functions of τ. We do not specify how the relations (2.5) could be obtained by a limiting operation from a particle of infinitesimal size, but we suppose that a spherical particle results from the choice $I_{ij}=I\delta_{ij}$.

A mechanical spin angular momentum four-vector s^{μ}_{mech} is required, and this is defined in a like manner to ω^{μ}. A mechanical spin tensor is also defined:

$$s^{\mu}_{\text{mech}} \stackrel{\text{def}}{=} s^{\text{mech}}_{(i)} u^{\mu}_{(i)}, \qquad s^{\mu\nu}_{\text{mech}} \stackrel{\text{def}}{=} -c\varepsilon^{\mu\nu\lambda\tau} s^{\text{mech}}_{\lambda} \dot{x}_{\tau}. \tag{2.6}$$

2.2. *The Lagrangian*

The Lagrangian equations arising from the following Lagrangian have been derived previously and are reproduced in § A1.3. The Lagrangian is chosen so that the free spinning particle equations (1.1) can be derived as Lagrangian equations in a general way using moments of inertia. The Lagrangian model automatically leads to the free particle equations (1.1):

$$L=-\tfrac{1}{2}m_0c^2(\dot{x}^{\mu}\dot{x}_{\mu}-1)+\tfrac{1}{2}s^{\text{mech}}_{(i)}\omega_{(i)}+\lambda_{0i}\dot{x}^{\mu}u_{(i)\mu}+\tfrac{1}{2}\lambda_{ij}(u^{\mu}_{(i)}u_{(j)\mu}+\delta_{ij}). \tag{2.7}$$

The kinetic energy terms are chosen on the basis of similarity with classical mechanics, and the second term representing the spin-energy assumes the use of (2.2) and (2.5). The condition $\dot{x}^{\mu}\dot{x}_{\mu}=1$ and the relations (2.1) are enforced in the Lagrangian formalism by the use of the scalar invariant Lagrange multipliers m_0, λ_{0i}, λ_{ij}, where the multiplier m_0 is taken without loss of generality to be the rest mass of the particle; it is further understood that λ_{ij} is identical to λ_{ji}. The use of Lagrange multipliers is *essential*, since no equivalent formalism based on a Lagrangian homogeneous of the first order in the generalised velocities exists.

The 26 canonical coordinates are x^{μ}, $u^{\mu}_{(i)}$, m_0, λ_{0i}, $\lambda_{ij}(\equiv\lambda_{ji})$. The 26 canonical momenta are defined as

$$\begin{aligned} p_{\mu} &\stackrel{\text{def}}{=} -\partial L/\partial\dot{x}^{\mu}, & \pi_{(i)\mu} &\stackrel{\text{def}}{=} -\partial L/\partial\dot{u}^{\mu}_{(i)}, & & \\ \Pi_0 &\stackrel{\text{def}}{=} -\partial L/\partial\dot{m}_0, & \Pi_{0i} &\stackrel{\text{def}}{=} -\partial L/\partial\dot{\lambda}_{0i}, & \Pi_{ij} &\stackrel{\text{def}}{=} -\partial L/\partial\dot{\lambda}_{ij}. \end{aligned} \tag{2.8}$$

The last expression is assumed to be defined only for the pairs of suffixes 2, 3; 3, 1; 1, 2; 1, 1; 2, 2; 3, 3 (the order of suffixes for λ_{ij} is immaterial), and we symmetrise by defining $\Pi_{ij}=\Pi_{ji}$. We introduce the following combinations of canonical variables:

$$s_{(i)} \stackrel{\text{def}}{=} -c^{-1}\varepsilon_{ijk}\pi^{\sigma}_{(j)}u_{(k)\sigma}, \qquad s^{\mu\nu} \stackrel{\text{def}}{=} 2\pi^{[\mu}_{(i)}u^{\nu]}_{(i)}, \tag{2.9}$$

here called the *internal components of canonical spin*, and the *canonical spin tensor*, respectively.

When the canonical momenta (2.8) and the combinations (2.9) are evaluated for the Lagrangian (2.7) (§ A1.3) the following values are found:

$$s_{(i)}=s_{(i)}^{\text{mech}}, \qquad s^{\mu\nu}=s^{\mu\nu}_{\text{mech}}, \qquad p^{\mu}=m_0c^2\dot{x}^{\mu}+s^{\mu\nu}_{\text{mech}}\ddot{x}_{\nu} \overset{\text{def}}{=} p^{\mu}_{\text{mech}}. \tag{2.10}$$

The last value is the mechanical momentum for the equations (1.1). The first two values are the internal components of mechanical spin and the mechanical spin tensor given in (2.5) and (2.6), respectively. Thus the canonical values are the same as the mechanical ones, and the spin tensor automatically satisfies the Frenkel condition of spin $s^{\mu\nu}\dot{x}_{\nu}=0$ by virtue of (2.6). The second-order Lagrange equations give rise to the free spinning particle equations (1.1):

$$\dot{p}^{\mu}=0, \qquad \dot{s}^{\mu\nu}+2p^{[\mu}\dot{x}^{\nu]}=0, \tag{2.11}$$

and the first-order Lagrange equations ensure that the constraints (2.1) are satisfied, and ensure the unit norm of $\dot{x}^{\mu}$.

The structure we have given to the tensor $s^{\mu\nu}_{\text{mech}}$ manifests itself in the Lagrangian formalism in the derivation of the equations

$$\dot{s}_{(i)}+c^{-1}\varepsilon_{ijk}\omega_{(j)}s_{(k)}=0, \tag{2.12}$$

representing the constancy of the spin angular momentum in the particle's rest frame, leading (for constant Is) to Euler's equations for the system.

3. The canonical constraints

3.1. The definition of the Poisson bracket

When the spin-energy term is represented in terms of the u's and the $\dot{u}$'s by the expressions (2.2) and (2.5), the Lagrangian L given by (2.7) is a function of 26 coordinates (as above) and the 16 velocities $\dot{x}^{\mu}$, $\dot{u}^{\mu}_{(i)}$. The m_0 and the λ's are 10 coordinate multipliers, and there is an explicit dependence on τ through I_{ij}. The 26 canonical momenta have already been defined by (2.8).

We shall require certain combinations of the canonical variables, including the combinations (2.9). These are:

$$\begin{aligned} cs_i &\overset{\text{def}}{=} \varepsilon_{ijk}u^{\sigma}_j\pi_{k\sigma}, & s^{\mu\nu} &\overset{\text{def}}{=} -2u_i^{[\mu}\pi_i^{\nu]}, \\ \lambda^{\mu} &\overset{\text{def}}{=} \lambda_{0i}u^{\mu}_i, & v^{\mu} &\overset{\text{def}}{=} (p^{\mu}+\lambda^{\mu})/m_0c^2, \qquad s^{\mu} \overset{\text{def}}{=} s_iu^{\mu}_i. \end{aligned} \tag{3.1a}$$

(We have now dropped brackets in suffixes, for convenience.) The combinations represented by v^{μ} and s^{μ} are values of the velocity and spin four-vectors, expressed in terms of the canonical variables; these values arise in the Lagrangian formalism (see § A1.3). The combinations represented by s_i and $s^{\mu\nu}$ are the internal components of spin and the spin tensor defined by (2.9), and already shown equal to the mechanical values. Based on the value found for $s^{\mu\nu}$ it can be shown that the following combinations of canonical variables represent the mechanical orbital and total angular momenta:

$$m^{\mu\nu} \overset{\text{def}}{=} -2x^{[\mu}p^{\nu]}, \qquad j^{\mu\nu}=m^{\mu\nu}+s^{\mu\nu}. \tag{3.1b}$$

The Poisson bracket of two quantities is defined in the usual way, and is with respect to *all* 26 pairs of canonical variables:

$$\{\xi,\eta\} \overset{\text{def}}{=} \left(\frac{\partial\xi}{\partial x^{\mu}}\frac{\partial\eta}{\partial p_{\mu}}-\frac{\partial\xi}{\partial p_{\mu}}\frac{\partial\eta}{\partial x^{\mu}}\right)+\left(\frac{\partial\xi}{\partial u_i^{\mu}}\frac{\partial\eta}{\partial \pi_{i\mu}}-\frac{\partial\xi}{\partial \pi_{i\mu}}\frac{\partial\eta}{\partial u_i^{\mu}}\right)$$
$$+\left(\frac{\partial\xi}{\partial m_0}\frac{\partial\eta}{\partial \Pi_0}-\frac{\partial\xi}{\partial \Pi_0}\frac{\partial\eta}{\partial m_0}\right)+\left(\frac{\partial\xi}{\partial \lambda_{0i}}\frac{\partial\eta}{\partial \Pi_{0i}}-\frac{\partial\xi}{\partial \Pi_{0i}}\frac{\partial\eta}{\partial \lambda_{0i}}\right)$$
$$+\left(\frac{\partial\xi}{\partial \lambda_{11}}\frac{\partial\eta}{\partial \Pi_{11}}-\frac{\partial\xi}{\partial \Pi_{11}}\frac{\partial\eta}{\partial \lambda_{11}}\right)+\left(\frac{\partial\xi}{\partial \lambda_{23}}\frac{\partial\eta}{\partial \Pi_{23}}-\frac{\partial\xi}{\partial \Pi_{23}}\frac{\partial\eta}{\partial \lambda_{23}}\right)$$
$$+\text{two similar terms} \qquad +\text{two similar terms.} \tag{3.2}$$

The space components of the four-vector represented by p_μ in (3.2) are the components of $-c\boldsymbol{p}$, where $\boldsymbol{p}$ is the three-momentum. Consequently this definition of the Poisson bracket will differ by a factor of $-c$ from the usual definition.

The Poisson brackets of the quantities representing position, momentum, and orbital, spin and total angular momentum can be found. The calculations for these are quite straightforward from the definition (3.2) and do not depend on the precise form of the Lagrangian (2.7). Because there are constraint equations in phase space, connecting the canonical variables, the definition (3.2) for the Poisson bracket must be modified in order that the correct equations of motion are obtained. The modified Poisson bracket is called the Dirac bracket, and details of these calculations are given in paper II.

3.2. The 'first-kind' primary constraints

Normally, in the derivation of the canonical equations arising from a Lagrangian, the Hessian matrix of the Lagrangian is non-singular, so that all the generalised velocities may be uniquely solved in terms of the canonical variables and the (proper) time. The *Hessian matrix* of the Lagrangian is the square matrix of second-order partial derivatives of the Lagrangian with respect to all the generalised velocities. In the present problem the Hessian matrix of the Lagrangian (2.7) for a free particle, with respect to all the generalised velocities, is 26-dimensional and singular of rank at most 16, when the spin-energy term is represented by the expressions (2.2) and (2.5). We therefore use the degenerate formalism. Initially we use the formalism of Shanmugadhasan (1973) with $n=26$ and $r_1=10$.

Because the difference between the dimensionality and the rank of the Hessian matrix is at least 10, there are at least 10 subsidiary conditions expressing the functional dependences for the canonical variables that follow immediately from the singularity of the Hessian matrix . (These subsidiary conditions we shall call *first-kind* subsidiary conditions.) In the present problem these subsidiary conditions are immediately identified as the 10 zero momenta conjugate to the multipliers:

$$\phi_0 \overset{\text{def}}{=} \Pi_0 \approx 0, \qquad \phi_{0i} \overset{\text{def}}{=} \Pi_{0i} \approx 0, \qquad \phi_{ij} \overset{\text{def}}{=} \Pi_{ij} \approx 0, \tag{3.3}$$

where the special symbol of Dirac (1958), '$\approx$', denotes that they are *weak* equations, i.e. they must not be used *before* working out the Poisson brackets, defined in (3.2).

In the general theory for a degenerate Lagrangian there are r_1 first-kind conditions, where n and $(n-r_1)$ are respectively the dimension and rank of the Hessian matrix. The method has been explained by Shanmugadhasan (1973) in the section leading to his equation (11), p 680, and also by Mukunda (1976) in the work preceding equation (2.4), p 416.

3.3. *The 'second-kind' primary constraints*

In Shanmugadhasan's formalism it is shown that further subsidiary conditions on the canonical variables are required and these conditions cannot be obtained by using Dirac's theory of multipliers. Such subsidiary conditions arise from the first-order Lagrange equations (and will be called 'second-kind' conditions). In the case of our degenerate Lagrangian (2.7), which does not contain the velocities corresponding to the multipliers, the vanishing momenta (3.3) lead to first-order Lagrange equations that do not contain these velocities. When the velocities they do contain are substituted in terms of the canonical variables only, we obtain the second-kind conditions. These are the following 10 weak equations:

$$\phi_0' \stackrel{\text{def}}{=} m_0^2c^4(v^\mu v_\mu - 1) \approx 0, \qquad \phi_{0i}' \stackrel{\text{def}}{=} m_0c^2 v^\mu u_{i\mu} \approx 0, \qquad \phi_{ij}' \stackrel{\text{def}}{=} u_i^\mu u_{j\mu} + \delta_{ij} \approx 0. \quad (3.4)$$

The velocities $\dot{x}^\mu$ have been eliminated by using the value of p^μ derived from the Lagrangian:

$$p^\mu = m_0c^2\dot{x}^\mu - \lambda^\mu \quad (3.5)$$

(the formula reproduces our definition of v^μ in (3.1*a*) and is derived in § A1.3).

As far as the derivation of these 'second-kind' conditions is concerned, the general theory appears to follow two quite separate paths. In the resumé of the Dirac theory given by Mukunda (1976) the method for finding the second-kind conditions is not given. It is stated (p 419) that the process of using the multiplier rule 'continues and at the end of it, there will be a set of self-perpetuating constraints' (presumably including the ϕ''s). This is in essence the Dirac theory. On the other hand, Shanmugadhasan believes that the current theory of multipliers does not correctly handle the subsidiary conditions because all the subsidiary conditions must be known before setting up the multiplier rule and before the canonical theory is applied. Consequently the multiplier rule cannot be used to obtain these conditions. (His method for finding the r_2 ($\leqslant r_1$) second-kind conditions is given in the sections leading to his equation (13), p 680.)

It is quite clear that the constraints (3.4) cannot be determined from (3.3) using the current theory of multipliers, although the constraints (3.4) are necessary for the canonical formalism. We therefore use Shanmugadhasan's formulation and we anticipate that there may be further constraints arising from the derivatives of all these constraints.

3.4. *The secondary constraints*

In Shanmugadhasan's treatment there is an assumed limitation in the scope of the initial Lagrange problem and no further constraints on the canonical variables beyond the primary constraints arise. However, it can be shown that further constraints, in addition to primary ones, are required. Because of the formal similarity with Dirac's procedure for finding new constraints, we have called such constraints *secondary constraints* and denoted them by χ's, as in Dirac's theory, but they may not be deducible from the multiplier rule in general. Our procedure falls outside the scope of Shanmugadhasan's here, because his treatment is limited to those Lagrangians that satisfy the requirement that the time derivatives of the first-order Lagrange equations hold by virtue of all the undifferentiated Lagrange equations—the 'consistency conditions' on the Lagrangian. These conditions are not satisfied by the Lagrangian (2.7)†.

One method for finding these secondary constraints is to differentiate the first-order Lagrange equations and then to substitute for the velocities in terms of the coordinates and the momenta. This process of differentiation and substitution is continued until all the secondary constraints have been found. The use of the Lagrange equations in this process is also allowed.

However, though this method may be successful in certain cases for determining the secondary constraints, there is considerable difficulty in the present problem in expressing the velocities $\dot{u}_i^\mu$ in terms of the coordinates and the momenta via the complicated expressions that are found for the momenta (see § A1.3):

$$\pi_i^\mu = -\tfrac{1}{4}c^2 \varepsilon_{pin}\varepsilon_{qkl}I_{pq}u_n^\mu \dot{u}_k^\sigma u_{l\sigma}. \tag{3.6}$$

We have therefore used an alternative method. We have found it more convenient to use the relationships that the momenta (3.6) satisfy directly by virtue of the first-order Lagrange equations. Some or all of the secondary constraints must involve the π's since they have as yet been unused in any of the constraints. These relationships will include the nine constraints

$$\chi_{0i} \overset{\text{def}}{=} m_0 c^2 v^\mu \pi_{i\mu} \approx 0, \qquad \chi_{ij} \overset{\text{def}}{=} \pi_i^\mu u_{j\mu} + \pi_j^\mu u_{i\mu} \approx 0, \tag{3.7}$$

which follow from (3.6). However, this will not produce all of the secondary constraints.

The remaining secondary constraints may be found by taking into account the conditions on the *canonical* variables derived from the *Lagrange* equations, that do not involve proper time derivatives (we consider how many combinations of the 52 canonical variables may be chosen arbitrarily subject to such conditions). After some work, we find that only one further secondary constraint is required, and this is the unused constraint on the π's occurring in the Lagrangian equations:

$$\chi_0 \overset{\text{def}}{=} \lambda_{0i}s_i \approx 0. \tag{3.8}$$

All other similar conditions on the canonical variables derived from the Lagrange equations may be deduced directly from the 29 constraints (3.3), (3.4), (3.7) already found. (We mention that this does not apply to the precise values of the λ_{ij}'s occurring in the Lagrangian equations.)

The final secondary constraint (3.8) may also be determined by using the multiplier rule with the previous incomplete set of 29 constraint equations, so that this is in some sense a confirmation of it. We emphasise, however, that *not all* of the secondary constraints (3.7), (3.8) may be determined by using the multiplier rule with the set of 20 primary constraints (3.3), (3.4).

† Even where the Lagrangian is such that all of the derivatives of the first-order Lagrange equations hold by virtue of all the undifferentiated Lagrange equations, Shanmugadhasan (1973) has shown that the canonical multiplier rule cannot, in general, be used iteratively to generate all the constraints. Consequently we must rely on the initial Lagrangian formalism to produce the phase space constraints, and we use Shanmugadhasan's formalism to deduce the first two types of constraint given above, since we believe that this correctly interprets the theory. (The completeness of the constraints is, in any case, later checked by the once-and-for-all use of the multiplier rule, to deduce all the original Lagrange equations canonically, so the initial formalism is really an academic point, the main concern being the production of the complete set of constraints by some method or other.)

The above procedure is now confirmed by the use of the multiplier rule (see below) on the 30 constraint equations (now called the 'subsidiary conditions'), where it is verified that the derivatives of these conditions give rise to no further constraints beyond the constraints already found.

3.5. The Poisson brackets of the constraints

Dirac's canonical theory involves the use of the multiplier rule in a restricted form, where fewer constraints arise in the canonical equations. The 'secondary constraints' are assumed not to arise in the total Hamiltonian. According to Dirac's theory, the consistency conditions (see below) arising from the time derivatives of the original constraints lead both to equations for the multipliers and to new equations between the canonical variables. In the latter case they are regarded as new constraint equations. In comparison with this treatment, the multiplier rule cannot here be used to generate new subsidiary conditions from the given ones, as in Dirac's theory, since we are not given beforehand a set of (primary) subsidiary conditions which are known to be the only ones arising in the canonical equations of motion. All the subsidiary conditions are equally likely to appear in the total Hamiltonian.

We later confirm that the set of 30 constraints (3.3), (3.4), (3.7), (3.8) forms a complete set. We follow Shanmugadhasan's general theory and make no distinction between primary and secondary constraints. We treat the constraints on an equal footing. We here calculate the 30-dimensional matrix of Poisson brackets (PBS), using the constraints to simplify the brackets only *after* the differentiations have been performed. Thus the values of the PBS of the constraints with each other are given in the table below. (In II we compute the inverse of a submatrix of this matrix of PBS. In order to make this calculation easier we re-order the constraints and renormalise some of them, and this new arrangement is used in the table.)

With the use of the 30 constraints, we may construct further constraints, which may be used to replace any or all of the 30 in any different formulation of the problem. These constraints, functionally dependent on the 30 given ones, are used later, after setting up the multiplier rule. We have given some of these in appendix 2. Such equivalent constraints include, for example, the Frenkel condition $s^{\mu\nu}v_\nu \approx 0$ in terms of the definitions (3.1*a*).

The only first-class constraints are the six constraints $\phi_{ij} \approx 0$, which are the vanishing momenta conjugate to the multipliers λ_{ij}. These are the only constraints adjacent to rows (or columns) of zeros. (First-class constraints are the constraints that correspond to rows (or columns) of zeros in the (r_3-dimensional) matrix of mutual PBS of the constraints. They have no effect on the equations of motion. Other rows and columns form the non-singular submatrix of PBS of the second-class constraints of even dimensionality (r_4).)

4. The derivation of Hamilton's equations

4.1. The canonical multipliers

The following argument now traces Shanmugadhasan's multiplier rule as it is applied to the (complete) set of 30† constraints (3.3), (3.4), (3.7), (3.8). The Hamiltonian for the

† We have included the first-class constraints, although the corresponding multipliers will remain undetermined and may be set to zero.

Table 1. Poisson brackets of the constraints.

	$c^{-2}\phi_0\}$	$\frac{1}{2}\phi'_0\}$	$c\chi_0\}$	$\phi_{0k}\}$	$\phi'_{0k}\}$	$\chi_{0k}\}$	$\phi_{kl}\}$	$\phi'_{kl}\}$	$\chi_{kl}\}$
$\{c^{-2}\phi_0,$	0	m_0c^2	0	0	0	0	0	0	0
$\{\frac{1}{2}\phi'_0,$	$-m_0c^2$	0	0	0	0	$m_0^2c^4\lambda_{0k}$	0	0	0
$\{c\chi_0,$	0	0	0	cs_k	0	0	0	0	0
$\{\phi_{0i},$	0	0	$-cs_i$	0	δ_{ik}	$-\frac{1}{2}\varepsilon_{ikp}cs_p$	0	0	0
$\{\phi'_{0i},$	0	0	0	$-\delta_{ki}$	0	$m_0^2c^4\delta_{ik}$	0	0	$(-\delta_{ik}\lambda_{0l}-\delta_{il}\lambda_{0k})$
$\{\chi_{0i},$	0	$-m_0^2c^4\lambda_{0i}$	0	$\frac{1}{2}\varepsilon_{kip}cs_p$	$-m_0^2c^4\delta_{ki}$	0	0	0	$(-\frac{1}{2}\varepsilon_{ikp}\lambda_{0l}cs_p-\frac{1}{2}\varepsilon_{ilp}\lambda_{0k}cs_p)$
$\{\phi_{ij},$	0	0	0	0	0	0	0	0	0
$\{\phi'_{ij},$	0	0	0	0	0	0	0	0	$(-2\sigma_{ik}\delta_{lj}-2\delta_{jk}\delta_{li})$
$\{\chi_{ij},$	0	0	0	0	$(\delta_{ki}\lambda_{0j}+\delta_{kj}\lambda_{0i})$	$(\frac{1}{2}\varepsilon_{kip}\lambda_{0j}cs_p+\frac{1}{2}\varepsilon_{kjp}\lambda_{0i}cs_p)$	0	$(2\delta_{ki}\delta_{jl}+2\delta_{li}\delta_{jk})$	$(-\delta_{ik}\varepsilon_{jlp}cs_p-\delta_{il}\varepsilon_{jkp}cs_p$ $-\delta_{jl}\varepsilon_{ikp}cs_p-\delta_{jk}\varepsilon_{ilp}cs_p)$

motion is written in the form

$$H = H_0 + \mu_0\phi_0 + \mu_{0i}\phi_{0i} + \tfrac{1}{2}\mu_{ij}\phi_{ij} + \mu'_0\phi'_0 + \mu'_{0i}\phi'_{0i} + \tfrac{1}{2}\mu'_{ij}\phi'_{ij} + \nu_0\chi_0 + \nu_{0i}\chi_{0i} + \tfrac{1}{2}\nu_{ij}\chi_{ij}, \quad (4.1)$$

where the μ's, μ''s and ν's are arbitrary multiplier functions to be determined. (The functions μ_{ij}, μ'_{ij}, ν_{ij} are assumed, without loss of generality, to be symmetric in the suffixes i, j.) The Hamiltonian H_0 is found in the usual way, normally by the use of the first-kind conditions only, but this restriction is not necessary. We may use any of the constraints to simplify it:

$$H_0 \approx -(p^\mu \dot{x}_\mu + \pi_i^\mu \dot{u}_{i\mu} + \Pi_0 \dot{m}_0 + \Pi_{0i}\dot{\lambda}_{0i} + \Pi_{11}\dot{\lambda}_{11} + \ldots + \Pi_{23}\dot{\lambda}_{23} + \ldots) - L$$

(for an explanation of the signs, see, for example, Mann (1974, p 127))

$$\approx -m_0c^2 - \pi_i^\mu \dot{u}_{i\mu} - L$$

$$\approx -m_0c^2 - \tfrac{1}{2}\pi_i^\mu \dot{u}_{i\mu}.$$

This last expression is obtained by the use of Euler's theorem on the only term of the Lagrangian that is weakly non-zero, this term being of second order in the $\dot{u}$'s. (This is equivalent to the use of the relations (3.6).) Using the expression (3.6) for the π's in the definition (3.1a) for s_i, we find the further weak equality $I_{pq}^{-1}s_ps_q \approx -\pi_i^\mu \dot{u}_{i\mu}$, in which we have assumed that the symmetric inertia matrix is non-singular and possesses an inverse given by $I_{ij}^{-1}I_{jk} = \delta_{ik}$. Hence the Hamiltonian finally reduces to the following simplified expression by the use of the constraints:

$$H_0 \overset{\text{def}}{=} -m_0c^2 + \tfrac{1}{2}I_{pq}^{-1}s_ps_q. \quad (4.2)$$

In order to evaluate the multipliers in (4.1), we require the PBs of H_0 with the constraints (simplified by using the constraints if necessary). For convenience in writing these expressions we introduce the following notation (cf (2.5)):

$$\omega_p \overset{\text{def}}{=} I_{pq}^{-1}s_q (= -c^{-1}\varepsilon_{qrs}I_{pq}^{-1}\pi_r^\sigma u_{s\sigma}).$$

We confirm later (when we have derived the Hamilton equations) that these are the components of the angular velocity (2.2) expressed in terms of the canonical variables. Using this notation we list the PBs of H_0 with the constraints, that are non-zero (we allow the constraints to simplify the brackets if necessary).

$$\{\phi_0, H_0\} = c^2, \qquad \{\phi'_{0i}, H_0\} \approx c^{-1}\varepsilon_{ipn}\omega_p\lambda_{0n},$$
$$\{\chi_0, H_0\} = -c^{-1}\varepsilon_{ipn}s_i\omega_p\lambda_{0n}, \qquad \{\chi_{0i}, H_0\} \approx \tfrac{1}{2}s_p\omega_p\lambda_{0i}. \quad (4.3)$$

These brackets are required in the 'consistency conditions' (below), and with these equations we are in a position to work out most of the values of the multiplier functions arising in the total Hamiltonian H.

4.2. The consistency conditions and the determination of the multipliers

The canonical equation of motion for any function g of coordinates, momenta and proper time is

$$\mathrm{d}g/\mathrm{d}\tau = \partial g/\partial\tau - \{g, H\} \approx \partial g/\partial\tau - \{g, H_0\} - \mu_0\{g, \phi_0\} - \mu_{0i}\{g, \phi_{0i}\} - \ldots. \quad (4.4)$$

When the subsidiary conditions (3.3), (3.4), (3.7), (3.8) are substituted for g in turn, and the results set to zero, we deduce the values for the multipliers. These equations are the 'consistency conditions', and they ensure that the proper time derivatives of all the constraints vanish. Using the values of the mutual PBs of the constraints given in table 1, and the values of the PBs of H_0 with the constraints given in (4.3), we find the following 30 consistency conditions:

$$\dot{\phi}_0 \approx -c^2 - 2m_0c^4\mu'_0 = 0,$$
$$\dot{\phi}_{0i} \approx -\mu'_{0i} + s_i\nu_0 + \tfrac{1}{2}c\varepsilon_{ikp}s_p\nu_{0k} = 0,$$
$$\dot{\phi}_{ij} \approx 0,$$
$$\dot{\phi}'_0 \approx 2m_0c^4\mu_0 - 2m_0^2c^4\lambda_{0k}\nu_{0k} = 0,$$
$$\dot{\phi}'_{0i} \approx -c^{-1}\varepsilon_{ipn}\omega_p\lambda_{0n} + \mu_{0i} - m_0^2c^4\nu_{0i} = 0,$$
$$\dot{\phi}'_{ij} \approx 2\nu_{ij} = 0,$$
$$\dot{\chi}_0 \approx c^{-1}\varepsilon_{ipn}s_i\omega_p\lambda_{0n} - s_k\mu_{0k} = 0,$$
$$\dot{\chi}_{0i} \approx -\tfrac{1}{2}s_p\omega_p\lambda_{0i} - \tfrac{1}{2}c\varepsilon_{kip}s_p\mu_{0k} + 2m_0^2c^4\lambda_{0i}\mu'_0 + m_0^2c^4\mu'_{0i} = 0,$$
$$\dot{\chi}_{ij} \approx -(\lambda_{0i}\mu'_{0j} + \lambda_{0j}\mu'_{0i}) - \tfrac{1}{2}c(\varepsilon_{kip}\lambda_{0j} + \varepsilon_{kjp}\lambda_{0i})s_p\nu_{0k} - 2\mu'_{ij} = 0.$$

We have used the values of some of the ν's from the sixth equation in simplifying some of the other equations. On using the subsidiary conditions, we have the following unique solution for 24 of the multipliers, in which it is assumed that m_0 and s_is_i are non-vanishing (we have omitted the details of the derivation):

$$\mu_0 \approx 0, \qquad \mu_{0i} \approx c^{-1}\varepsilon_{ipn}[\omega_p + m_0c^2(s_ks_k)^{-1}s_p]\lambda_{0n},$$
$$\mu'_0 = -\tfrac{1}{2}m_0^{-1}c^{-2}, \qquad \mu'_{0i} \approx \tfrac{1}{2}m_0^{-1}c^{-2}\lambda_{0i}, \qquad \mu'_{ij} \approx 0, \qquad \nu_0 = 0, \qquad \nu_{ij} = 0, \tag{4.5}$$
$$\nu_{0i} \approx m_0^{-1}c^{-3}(s_ks_k)^{-1}\varepsilon_{ipn}s_p\lambda_{0n} \approx m_0^{-1}c^{-2}(\pi^{\tau}_k\pi_{k\tau})^{-1}p^{\sigma}\pi_{i\sigma}. \tag{4.6}$$

The six multipliers μ_{ij} remain undetermined since the constraints ϕ_{ij} are first class. The rest of the multipliers have been determined by successive elimination using the second-class constraints. In this elimination the intermediate results

$$\lambda_{0i} \approx -m_0c^3\varepsilon_{ipk}s_p\nu_{0k}, \qquad s_i\nu_{0i} \approx 0 \tag{4.7}$$

are found, and these lead to the values of the ν's given in (4.6). The alternative expression in (4.6) has been derived from relations given in appendix 2.

Thus, of the 30 subsidiary conditions we started with, six are first class, leaving an even number of second-class constraints. The first-class constraints have no effect on the equations of motion; and the number of independent second-class constraints must be an even number for the multiplier rule to apply, otherwise these second-class constraints are not independent (Dirac 1964).

4.3. The total Hamiltonian and Hamilton's equations

With the explicit values (4.5), (4.6) for the multipliers, the total Hamiltonian reduces to

$$H = -m_0c^2 + \tfrac{1}{2}\omega_ps_p + \mu_{0i}\phi_{0i} + \mu'_0\phi'_0 + \mu'_{0i}\phi'_{0i} + \nu_{0i}\chi_{0i}$$
$$= -\tfrac{1}{2}m_0c^2 + \tfrac{1}{2}\omega_ps_p + \mu_{0i}\Pi_{0i} - \tfrac{1}{2}m_0^{-1}c^{-2}(p^{\mu} + \lambda_{0i}u_i^{\mu})(p_{\mu} - 2m_0c^2\nu_{0j}\pi_{j\mu}). \tag{4.8}$$

We omit the first-class constraints since the equations arising from the λ_{ij} and the Π_{ij} are not needed. This conforms with Shanmugadhasan's treatment in which the multipliers corresponding to the constraints that do not modify the canonical equations are set to zero.

The canonical covariant Hamilton equations for the total Hamiltonian (4.8) arise from the same canonical equations (4.4) by substituting $g = x^\mu, u_i^\mu, m_0, \lambda_{0i}, p^\mu, \pi_i^\mu, \Pi_0, \Pi_{0i}$ in turn. When we adjoin the subsidiary conditions (3.3), (3.4), (3.7), (3.8), these equations should give the correct equations derivable from the Lagrangian equations. These Hamilton equations are

$$\dot{x}^\mu = -\partial H/\partial p_\mu \approx m_0^{-1} c^{-2}(p^\mu + \lambda^\mu), \tag{4.9}$$

$$\dot{u}_i^\mu = -\partial H/\partial \pi_{i\mu} = -\nu_{0i}(p^\mu + \lambda^\mu) - c^{-1}\varepsilon_{ijk}\omega_j u_k^\mu, \tag{4.10}$$

$$\dot{m}_0 = -\partial H/\partial \Pi_0 = 0, \tag{4.11}$$

$$\dot{\lambda}_{0i} = -\partial H/\partial \Pi_{0i} = -m_0^2 c^4 \nu_{0i} - c^{-1}\varepsilon_{ijk}\omega_j \lambda_{0k}, \tag{4.12}$$

$$\dot{p}^\mu = \partial H/\partial x_\mu = 0, \tag{4.13}$$

$$\dot{\pi}_i^\mu = \partial H/\partial u_{i\mu} \approx -\tfrac{1}{2} m_0^{-1} c^{-2} \lambda_{0i}(p^\mu + \lambda^\mu) - c^{-1}\varepsilon_{ijk}\omega_j \pi_k^\mu, \tag{4.14}$$

$$\dot{\Pi}_0 = \partial H/\partial m_0 = \tfrac{1}{2} m_0^{-2} c^{-2} p^\mu (p_\mu + \lambda_\mu) - \tfrac{1}{2} c^2 \approx 0, \tag{4.15}$$

$$\dot{\Pi}_{0i} = \partial H/\partial \lambda_{0i} \approx -\tfrac{1}{2} m_0^{-1} c^{-2}(p^\mu + \lambda^\mu) u_{i\mu} \approx 0. \tag{4.16}$$

4.4. Verification of the Hamilton equations and the completeness of the constraints

We now show that the Hamilton equations (4.9)–(4.16), when taken with the subsidiary conditions (which, incidentally, include the Frenkel condition), are consistent and give rise to all the correct equations. These equations are the Hamilton equations for the free particle equations (1.1) in terms of coordinates and momenta. The equation (4.9) is the equation for the momentum (3.5), and (4.11) and (4.13) establish the constancy of m_0 and p^μ. Equation (4.10), by contraction with $u_{j\mu}$, leads us to identify ω_i with the components of angular velocity (2.2); and (4.10) then represents (2.4*b*) in which

$$\dot{u}_i^\mu \dot{x}_\mu = -m_0 c^2 \nu_{0i}. \tag{4.17}$$

Likewise

$$\dot{\pi}_i^\mu \dot{x}_\mu = -\tfrac{1}{2}\lambda_{0i} \tag{4.18}$$

from (4.14). The equations (4.15) and (4.16) merely indicate that the derivatives of the second-class constraints of (3.3) vanish as expected. From all these equations and from equations (4.7), which follow from the values of the ν's, we verify that the derivatives of the first-order Lagrange equations and of the secondary constraints are satisfied. These lead to the equations

$$\dot{\lambda}^\mu \dot{x}_\mu = \lambda^\mu \ddot{x}_\mu = \dot{\lambda}^\mu \lambda_\mu = \dot{\lambda}_{0i}\lambda_{0i} = 0.$$

The derivation of the equations (2.11), (2.12) is obtained as follows. The equations (4.18) are used in (4.9) to find p^μ:

$$p^\mu = m_0 c^2 \dot{x}^\mu - 2\pi_i^\nu u_i^\mu \ddot{x}_\nu = m_0 c^2 \dot{x}^\mu - 2\pi_i^{[\nu} u_i^{\mu]} \ddot{x}_\nu = m_0 c^2 \dot{x}^\mu + s^{\mu\nu}\ddot{x}_\nu,$$

and the equations (2.12) may be obtained directly from (4.10) and (4.14) using the definition of s_i. The equations (2.11), subject to the Frenkel condition, result from the same equations using the definition of $s^{\mu\nu}$ and the weak equation (A2.4):

$$\dot{s}^{\mu\nu} = 2\dot{\pi}_i^{[\mu} u_i^{\nu]} + 2\pi_i^{[\mu}\dot{u}_i^{\nu]} = -2m_0^{-1}c^{-2}p^{[\mu}\lambda^{\nu]} = -2p^{[\mu}\dot{x}^{\nu]}.$$

Thus we have verified that the total Hamiltonian (4.8) is correct for the model, and we have incidentally verified that we have a complete set of constraints. The omission of any of these constraints would have severe consequences in the evaluation of Dirac brackets. The calculation of these brackets, based on the complete set of constraints, is given in II.

5. Conclusion

We have derived the complete set of canonical constraints and the covariant Hamilton equations for the Lagrangian for a classical particle with spin. We have initially used Dirac's theory of canonical multipliers as modified by Shanmugadhasan (1973) to derive the first two types of phase space constraint. The precise method of deriving the constraints is not important, but the verification of their completeness is. We note that Dirac's theory cannot here be used iteratively (by the use of the multiplier rule) to create new constraints, and that Shanmugadhasan's modification is itself insufficient for the correction of this. The fact that the iterative use of the multiplier rule does not lead to the determination of all the constraints is illustrated by observing that not all of the constraints that we have called 'secondary' are deducible from the original ones by using the multiplier rule. The extra work that is necessary to derive them is not covered by the Dirac theory. (This is not a special situation and many instances may occur in Dirac's theory where the multiplier rule cannot be used to obtain all the constraints in an iterative manner.)

Three types of subsidiary condition are met. (This generalises Shanmugadhasan's treatment where only *two* types arise.) We have called these conditions: (i) first-kind conditions—arising directly from the singularity of the Lagrangian (e.g. the zero momenta conjugate to the multipliers); (ii) second-kind conditions—arising from the first-order Lagrange equations (explicitly or implicitly); (iii) secondary constraints—arising from the derivatives of the first-order Lagrange equations. In general, Dirac's multiplier rule cannot be used to deduce conditions (ii) from conditions (i), nor to deduce conditions (iii) from conditions (i) and (ii). It is surprising that despite many accounts of Dirac's method, only Shanmugadhasan's (1973) indicates this in the general theory, and his method for the construction of conditions (ii) has been put into practice in this derivation. Shanmugadhasan's treatment is limited to those Lagrangians that satisfy the requirement that the time derivatives of all the first-order Lagrange equations hold by virtue of all the undifferentiated Lagrange equations, so the special method for constructing conditions (iii) has been given.

Finally we note that the method of introducing extra constraints in Dirac's theory—known as 'Dirac gauge constraints' (Dirac 1958, 1959)—is an alternative to the method of Lagrange multipliers. These constraints need not be connected with the Lagrangian or the Lagrange equations but the use of these constraints is necessary. We believe that the method of Lagrange multipliers is a far better method because there is no doubt of its accuracy; it takes account of the momenta conjugate to the multipliers, which may be first- or second-class functions.

Appendix 1. The Lagrangian equations

In this appendix we give details concerning the manipulation of the quantities u_i^μ and their derivatives. In § A1.1 below, we derive (2.3) for the angular velocity four-vector, using the equivalent expressions representing the orthonormality relations (2.1). In § A1.2 we derive (2.4*a*) for the rates of change of the u's in terms of the undifferentiated u's, which include the 'extra' (non-classical) terms. In § A1.3 we derive the Lagrangian equations from the Lagrangian (2.7) independently of the previous calculations.

A1.1

The relations $\varepsilon_{ijk}u_{k\sigma} = \varepsilon_{\sigma\lambda\nu\tau}u_i^{\lambda}u_j^{\nu}\dot{x}^{\tau}$ express (2.1) in a tensor form analogous to $\boldsymbol{i} = \boldsymbol{j} \wedge \boldsymbol{k}, \ldots$ for the moving frame. Using these and (2.2), we find that

$$\omega_i = -\tfrac{1}{2}c\varepsilon_{\sigma\lambda\nu\tau}\dot{u}_j^{\sigma}u_i^{\lambda}u_j^{\nu}\dot{x}^{\tau}, \qquad \omega^{u} \overset{\text{def}}{=} \omega_i u_i^{\mu} = -\tfrac{1}{2}c\varepsilon_{\sigma\lambda\nu\tau}\dot{u}_j^{\sigma}(-g^{\lambda\mu})u_j^{\nu}\dot{x}^{\tau}.$$

(We have used the result (A2.2) expressing the orthonormality relations.) The result (2.3) then follows.

A1.2

In order to remove the rates of change $\dot{u}_{i\lambda}$ from the expression (2.3), we use (2.3) and the derivatives of the relations (2.1) to find an alternative form for the following:

$$\begin{aligned} c^{-1}\varepsilon^{\mu\nu\alpha\beta}\omega_\alpha\dot{x}_\beta &= -\tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\alpha\lambda\sigma\tau}\dot{u}_i^{\lambda}u_i^{\sigma}\dot{x}^{\tau}\dot{x}_\beta \\ &= -\dot{x}_\beta\dot{u}_i^{\beta}u_i^{[\mu}\dot{x}^{\nu]} - \dot{x}_\beta\dot{x}^{\beta}\dot{u}_i^{[\mu}u_i^{\nu]} \\ &= -\ddot{x}^{[\mu}\dot{x}^{\nu]} - \dot{u}_i^{[\mu}u_i^{\nu]}. \end{aligned} \tag{A1.1}$$

In deriving this result, we have used (A2.2) and the generalised Krönecker delta value $\delta^{\beta\mu\nu}_{\lambda\sigma\tau}$ in terms of Krönecker delta symbols, for the contraction of ε-symbols. Contracting the second term of the expression (A1.1) with $u_{i\nu}$ leads to

$$-\dot{u}_j^{[\mu}u_j^{\nu]}u_{i\nu} = \tfrac{1}{2}\dot{u}_j^{\mu}\delta_{ij} - \tfrac{1}{2}u_j^{\nu}u_j^{\mu}\dot{u}_{i\nu} = \dot{u}_i^{\mu} - \ddot{x}^{[\mu}\dot{x}^{\nu]}u_{i\nu},$$

where we have again used (A2.2) and also the orthonormality relations and their derivatives. Hence

$$\dot{u}_i^{\mu} = \Omega^{\mu\nu}u_{i\nu}, \qquad \Omega^{\mu\nu} = \ddot{x}^{[\mu}\dot{x}^{\nu]} - \dot{u}_i^{[\mu}u_i^{\nu]}.$$

Using the complete result (A1.1) we therefore find that $\Omega^{\mu\nu}$ has the value given in (2.4*a*). The expression (2.4*b*) is found by using the definition of ω^{μ} and the previous relations in § A1.1.

A1.3

It is perfectly possible to use the relations involving the ε-symbol in § A1.1 in a Lagrangian formalism with multipliers, in place of the 10 orthonormality relations. The advantage in doing this would be that we would have no need to assume a right-handed ordering of the axes. However, such a formalism (with the three-suffix alternating symbol in these relations removed to the right-hand side of the equation) would require the use of 12 multipliers, not all of which would be independent. This would unduly complicate the formalism, so we have instead used the simpler Lagrangian (2.7), which contains 10 independent multipliers and which does not imply a right-handed ordering of the axes. These Lagrangian equations have been given previously (Ellis 1975a).

The Lagrangian (2.7) is expressed in terms of the coordinates and velocities by using (2.2) and (2.5) to convert the spin-energy term into an expression containing these variables only. The first-order Lagrange equations for the variables m_0, λ_{0i}, and $\lambda_{ij}(\equiv\lambda_{ji})$ are the orthonormality relations $\dot{x}^\mu\dot{x}_\mu=1$ and (2.1). We need not assume a right-handed ordering of the set of four axes x^μ, u_i^μ until the identification of the canonical spin components with the mechanical ones is required. The equations (2.11) and (2.12) are obtained as follows. The second-order Lagrange equations are the 16 equations

$$\dot{p}^\mu+\partial L/\partial x_\mu=0, \qquad \dot{\pi}_i^\mu+\partial L/\partial u_{i\mu}=0, \tag{A1.2}$$

where the p's and π's are the generalised momenta corresponding to the x's and u's. The derivatives in (A1.2) are now listed:

$$\partial L/\partial x_\mu=0, \qquad \partial L/\partial u_{i\mu}=-\tfrac{1}{4}c^2\varepsilon_{pim}\varepsilon_{qkl}I_{pq}\dot{u}_m^\mu\dot{u}_k^\sigma u_{l\sigma}+\lambda_{0i}\dot{x}^\mu+\lambda_{ij}u_j^\mu,$$

$$p^\mu\overset{\text{def}}{=}-\partial L/\partial\dot{x}_\mu=m_0c^2\dot{x}^\mu-\lambda_{0i}u_i^\mu, \tag{A1.3}$$

$$\pi_i^\mu\overset{\text{def}}{=}-\partial L/\partial\dot{u}_{i\mu}=-\tfrac{1}{4}c^2\varepsilon_{pin}\varepsilon_{qkl}I_{pq}u_n^\mu\dot{u}_k^\sigma u_{l\sigma}.$$

The first of the equations (A1.2) immediately gives

$$\dot{p}^\mu=0. \tag{A1.4}$$

Certain components of the second may be evaluated by defining various combinations of (A1.3) as in (2.9). We have the following equations (the orthonormality relations and their derivatives have been used in only the first of these):

$$\begin{aligned}\dot{s}_r&=-c^{-1}\varepsilon_{rij}[\pi_i^\mu\dot{u}_{j\mu}-(\partial L/\partial u_{i\mu})u_{j\mu}]\\&=(\delta_{rn}\delta_{jp}-\delta_{rp}\delta_{jn})u_n^\mu\dot{u}_{j\mu}I_{pq}\omega_q\\&=c^{-1}\varepsilon_{rps}I_{pq}\omega_q\omega_s,\end{aligned} \tag{A1.5}$$

$$\dot{s}^{\mu\nu}=2\pi_i^{[\mu}\dot{u}_i^{\nu]}-2(\partial L/\partial u_{i[\mu})u_i^{\nu]}=2\lambda_{0i}u_i^{[\mu}\dot{x}^{\nu]}=-2p^{[\mu}\dot{x}^{\nu]}, \tag{A1.6}$$

where the symmetry of λ_{ij} has been used in both (A1.5) and (A1.6), and the expression (A1.5) uses the definition of the components of the angular velocity (2.2). The identification of the canonical spin components with the mechanical ones is made from the values of the last set of derivatives in (A1.3), as follows:

$$s_r=-c^{-1}\varepsilon_{rij}\pi_i^\mu u_{j\mu}=-\tfrac{1}{2}c\delta_{rp}I_{pq}\varepsilon_{qkl}\dot{u}_k^\sigma u_{l\sigma}=I_{rq}\omega_q=s_r^{\text{mech}}, \tag{A1.7}$$

$$s^{\mu\nu}=2\pi_i^\mu u_i^\nu=-c\varepsilon^{\mu\nu\lambda\tau}I_{pq}\omega_q u_{p\lambda}\dot{x}_\tau=-c\varepsilon^{\mu\nu\lambda\tau}s_\lambda^{\text{mech}}\dot{x}_\tau=s_{\text{mech}}^{\mu\nu}. \tag{A1.8}$$

In (A1.7) we have used the orthonormality relations and the definitions (2.2), (2.5), and in (A1.8) we have used the 'oriented' relations and the definitions (2.2), (2.5), (2.6).

Finally, we calculate the four-momentum given by (A1.3). This requires the values of the multipliers λ_{0i}, which are obtained by contracting the second of equations (A1.2) with $\dot{x}_\mu$:

$$\lambda_{0i}=-\dot{\pi}_i^\mu\dot{x}_\mu+\tfrac{1}{4}c^2\varepsilon_{pim}\varepsilon_{qkl}I_{pq}\dot{u}_m^\mu\dot{u}_k^\sigma u_{l\sigma}\dot{x}_\mu=2\pi_i^\mu\ddot{x}_\mu. \tag{A1.9}$$

(We have taken account of the dependence of I_{pq} on τ.) We find that

$$p^\mu=m_0c^2\dot{x}^\mu-2\pi_i^\nu u_i^\mu\ddot{x}_\nu=m_0c^2\dot{x}^\mu-2\pi_i^{[\nu}u_i^{\mu]}\ddot{x}_\nu=m_0c^2\dot{x}^\mu+s^{\mu\nu}\ddot{x}_\nu. \tag{A1.10}$$

We have thus deduced the free spinning particle equations (A1.4), (A1.6), for which the internal components of canonical spin and the canonical spin tensor have the mechanical values (A1.7), (A1.8), and for which the canonical momentum is given by (A1.10).

Appendix 2. The second-class constraints

The 24 second-class constraints that we have used in the present problem are the constraints (3.4), (3.7), (3.8) and the constraints (3.3) excluding the third set of six. With the use of all of these constraints we may construct further constraints which may be used to replace some of them in any different formulation of the problem. These constraints are also used in the present problem to simplify PBs after differentiations have been performed and to simplify the variables in DBs before or after DBs are calculated.

The definitions (3.1*a*), in vector notation, are

$$\boldsymbol{s}=\boldsymbol{u}^{\sigma}\wedge\boldsymbol{\pi}_{\sigma},\qquad s^{\mu\nu}=-2\boldsymbol{u}^{[\mu}\cdot\boldsymbol{\pi}^{\nu]},\qquad \lambda^{\mu}=\boldsymbol{\lambda}\cdot\boldsymbol{u}^{\mu},$$
$$v^{\mu}=m_0^{-1}c^{-2}(p^{\mu}+\lambda^{\mu}),\qquad cs^{\mu}=\boldsymbol{s}\cdot\boldsymbol{u}^{\mu},$$

representing, in terms of canonical variables, respectively the internal components of spin, the spin tensor, the four-vector corresponding to the multipliers λ_{0i}, the four-velocity and the 'four-spin'. The three values of the canonical multipliers calculated from (4.6), $\boldsymbol{\nu}=m_0^{-1}c^{-2}|\boldsymbol{s}|^{-2}\boldsymbol{s}\wedge\boldsymbol{\lambda}$, are taken to be definitions. Using all the second-class constraints with the exception of those in (3.3), we have the following weak relationships:

$$s^{\mu\nu}v_{\nu}\approx 0,\qquad \lambda^{\mu}v_{\mu}\approx 0,\qquad p^{\mu}v_{\mu}\approx m_0c^2,\qquad p^{\mu}\boldsymbol{u}_{\mu}\approx\boldsymbol{\lambda},$$
$$p^{\mu}s_{\mu}\approx 0,\qquad p^{\mu}\lambda_{\mu}\approx|\boldsymbol{\lambda}|^2,\qquad p^{\mu}p_{\mu}+|\boldsymbol{\lambda}|^2\approx m_0^2c^4, \tag{A2.1}$$

all of which are easily verified. The Frenkel condition of spin, for example, is an alternative second-class constraint. The 10 second-class constraints (3.4) representing the orthonormality relations are directly equivalent to the 10 independent components contained in the equation

$$v^{\mu}v^{\nu}-\boldsymbol{u}^{\mu}\cdot\boldsymbol{u}^{\nu}\approx g^{\mu\nu}. \tag{A2.2}$$

This may be verified by contracting the equation with u's and v's. Using the further set of second-class constraints (3.7), we also find $\boldsymbol{\pi}^{\mu}\cdot\boldsymbol{u}^{\nu}+\boldsymbol{\pi}^{\nu}\cdot\boldsymbol{u}^{\mu}\approx 0$, which are again verified in a similar way by contraction. From this result we have $s^{\mu\nu}\approx 2\boldsymbol{\pi}^{\mu}\cdot\boldsymbol{u}^{\nu}$ and $\boldsymbol{\pi}^{\mu}\approx-\frac{1}{2}s^{\mu\nu}\boldsymbol{u}_{\nu}$. (The first arises from the definition of $s^{\mu\nu}$, and the second by the contraction of the first with $u_{i\nu}$ and by the use of (3.4).) Using the definition of s, we deduce from the second expression that

$$\boldsymbol{\pi}^{\mu}\approx-\tfrac{1}{2}\boldsymbol{s}\wedge\boldsymbol{u}^{\mu}. \tag{A2.3}$$

We now verify the further expression contained in (4.6) and the expression (4.7), used in the evaluation of some of the multipliers. From (A2.3) and the weak relationships given above, we find that

$$p^{\mu}\boldsymbol{\pi}_{\mu}\approx-\tfrac{1}{2}\boldsymbol{s}\wedge\boldsymbol{\lambda}=-\tfrac{1}{2}m_0c^2|\boldsymbol{s}|^2\boldsymbol{\nu}\approx m_0c^2(\boldsymbol{\pi}^{\mu}\cdot\boldsymbol{\pi}_{\mu})\boldsymbol{\nu}.$$

This verifies the second expression of (4.6). From the definition of $\boldsymbol{\nu}$, and by the use of (3.8), we find equivalently that $\boldsymbol{\lambda}\approx-m_0c^2\boldsymbol{s}\wedge\boldsymbol{\nu}$ where $\boldsymbol{s}\cdot\boldsymbol{\nu}\approx 0$, which verifies (4.7).

The following products of u's and π's are also required:

$$2u_i^\mu \pi_{j\mu} \approx \varepsilon_{ijk} c s_k, \qquad 4\pi_i^\mu \pi_{j\mu} \approx c^2 s_i s_j - |s|^2 \delta_{ij},$$

$$4\boldsymbol{\pi}^\mu \cdot \boldsymbol{\pi}^\nu \approx s^{\mu\alpha} s_\alpha{}^\nu \approx |s|^2 \boldsymbol{u}^\mu \cdot \boldsymbol{u}^\nu - c^2 s^\mu s^\nu.$$

Contracting the last result with $|s|^{-2} p_\nu$ and using the previous results yields

$$-2m_0c^2 \boldsymbol{\pi}^\mu \cdot \boldsymbol{\nu} \approx 2(s^{\sigma\tau} s_{\sigma\tau})^{-1} s^{\mu\alpha} s_\alpha{}^\nu p_\nu \approx \lambda^\mu. \tag{A2.4}$$

The first weak result of (A2.4) has been used in the covariant Hamilton equations. The second enables us to express $m_0^2 c^4$ weakly in terms of the p's and the s's, from (A2.1).

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